Def → def → lemma → Thm → proof → corollary

2.3 – Properties of Determinants; Cramer's Rule

Properties of Determinants

$2\ \det(A) \neq \det(2A)$

- For an $n \times n$ matrix A, $\det(kA) = k^n \det(A)$

- **Lemma 2.3.2** (prelude to a later result)
  If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then $\det(EB) = \det(E)\det(B)$.

- **Theorem 2.3.3** A square matrix A is invertible if and only if $\det(A) \neq 0$.

- **Theorem 2.3.4** If A and B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.

  $\det(A+B) \neq \det(A) + \det(B)$

- **Theorem 2.3.5** If A is invertible, then $\det\left(A^{-1}\right) = \dfrac{1}{\det(A)}$.

1st bullet - one factor of k for each row

Note: For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$

$\begin{vmatrix} 3 & 6 \\ 3 & 4 \end{vmatrix} = 12 - 18 = -6 = 3(-2)$ as in 2.2

But $3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} \Rightarrow \det(3A) = \begin{vmatrix} 3 & 6 \\ 9 & 12 \end{vmatrix}$

$= 36 - 54 = -18$

$= 3^2(-2)$

Please note the difference in notation for a matrix vs. a determinant.

Lemma 2.3.2 – Suppose E results from multiplying the i^{th} row of I by k.

Then $EB = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vdots \\ k\vec{e}_i \\ \vdots \\ \vec{e}_n \end{bmatrix} B = \begin{bmatrix} \vec{e}_1 B \\ \vec{e}_2 B \\ \vdots \\ k\vec{e}_i B \\ \vdots \\ \vec{e}_n B \end{bmatrix}$, so the i^{th}

row of EB is the i^{th} row of B, multiplied by k. By Thm 2.2.3(a), $\det(E) = k$

so $\det(EB) = k \det(B) = \det(E)\det(B)$ ✓

Thm 2.3.3 $\quad$ $\text{rref}(A) = I_n$, $A\vec{x} = \vec{0}$ triv. $\quad$ Equivalent Statements

$A\vec{x} = \vec{b}$ has ! soln $\quad AA^{-1} = I$

($\Rightarrow$) A invertible $\Rightarrow$ A is expressible as a product of elementary matrices $\Rightarrow \det(A) \neq 0$.

($\Leftarrow$) $\det(A) \neq 0 \Rightarrow \text{rref}(A)$ does not have a row of zeros (Thm 2.2.1) $\Rightarrow \text{rref}(A) = I$

$\Rightarrow A$ is invertible. ✓

Thm 2.3.4 - If A and/or B is not invertible, then AB is not invertible and we have $0=0$.

If A & B are invertible, then we can use products of elementary matrices and Lemma 2.3.2.

Thm 2.3.5 - A invertible $\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

A invertible $\Rightarrow AA^{-1} = I$ and $\det(A) \neq 0$

$\Rightarrow \det(AA^{-1}) = \det(I)$

$\Rightarrow \det(A)\det(A^{-1}) = 1$

$\det(A^{-1}) = \frac{1}{\det(A)}$

✓

Theorem 2.3.8 Equivalent Statements (extends Theorem 1.6.4)

If A is an $n \times n$ matrix, then the following statements are equivalent.

same

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

g) $\det(A) \neq 0$.

new

#16 Find the values of k for which the matrix A is invertible.

$A = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix}$

A invertible $\Leftrightarrow$ $\det(A) \neq 0$

$\begin{vmatrix} k & 2 \\ 2 & k \end{vmatrix} = 0 \Rightarrow k^2 - 4 = 0 \Rightarrow k = \pm 2$

A is invertible for $k \neq \pm 2$.

#35 In each part, find the determinant given that A is a 3×3 matrix for which $\det(A) = 7$.

a. $\det(3A)$

b. $\det(A^{-1})$

c. $\det(2A^{-1})$

d. $\det((2A)^{-1})$

a) $\det(3A) = 3^3 \det A = 27(7) = 189$

b) $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{7}$

c) $\det(2A^{-1}) = 2^3 \det(A^{-1}) = \frac{8}{7}$

d) $\det((2A)^{-1}) = \frac{1}{2^3} \frac{1}{\det(A)} = \frac{1}{56}$

Definition: If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix $\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$ is called the **matrix of cofactors from A**. The transpose of this matrix is called the **adjoint of A** and is denoted by $\text{adj}(A)$.

Theorem 2.3.6 Inverse of a Matrix Using Its Adjoint

If A is an invertible matrix, then $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$.

#**20** Decide whether the matrix is invertible, and if so, use the adjoint to find its inverse.

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$

$$\det(A) = 2(-12) - 0(4) + 3(6) = -6$$

$\neq 0 \Rightarrow A$ is invertible

$M_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} = -12$ $\quad M_{12} = \begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} = 4$ $\quad M_{13} = \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6$

$C_{11} = -12$ $\quad C_{12} = -4$ $\quad C_{13} = 6$

$M_{21} = \begin{vmatrix} 0 & 3 \\ 0 & -4 \end{vmatrix}$ $\quad M_{22} = \begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix} = -2$ $\quad M_{23} = \begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix}$

$C_{21} = 0$ $\quad C_{22} = -2$ $\quad C_{23} = 0$

$M_{31} = \begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} = -9$ $\quad M_{32} = \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = 4$ $\quad M_{33} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$

$C_{31} = -9$ $\quad C_{32} = -4$ $\quad C_{33} = 6$

$$[C_{ij}] = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix} \Rightarrow [C_{ij}]^T = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

Cofactor matrix

$[C_{ij}]^T = \mathrm{adj}(A)$ the adjoint of A

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

$$A^{-1} = \frac{1}{-6} \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & 0 & 3/2 \\ 2/3 & 1/3 & 2/3 \\ -1 & 0 & -1 \end{bmatrix}$$

2×2: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad M_{11} = d \quad M_{12} = c$

$M_{21} = b \quad M_{22} = a$

$$[C_{ij}] = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \Rightarrow [C_{ij}]^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow \det(A) = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 2.3.7 Cramer's Rule

If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is $x_1 = \dfrac{\det(A_1)}{\det(A)}, \ x_2 = \dfrac{\det(A_2)}{\det(A)}, \ \ldots, \ x_n = \dfrac{\det(A_n)}{\det(A)}$ where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

#26 Solve by Cramer's rule.

$$x - 4y + z == 6$$

$$4x - y + 2z = -1$$

$$2x + 2y - 3z = -20$$

$$A = \begin{bmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{bmatrix}$$

$$x = \frac{\det(A_1)}{\det(A)}, \quad y = \frac{\det(A_2)}{\det(A}, \quad z = \frac{\det(A_3)}{\det(A)}$$